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Theory of the Z-mode Propagation

M. D. Papagiannis

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Harvard College Observatory
Cambridge 38, Massachusetts

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THEORY OF THE Z-MODE PROPAGATION

Notation

The following frequencies will appear in our formulae.

f = operating wave frequency (in Mc) $= \frac{\omega}{2\pi}$

$f_N = \left[\frac{Ne^2}{\pi m}\right]^{1/2} = 9 \times 10^3 (N)^{1/2}$ = plasma frequency (in Mc if N = el/cm^3)

$f_H = \frac{eH_o}{2\pi mc} = 2.8\ H_o$ = cyclotron frequency (in Mc if H_o in Gauss)

ν = collision frequency

$\nu_c = 2\pi f_H \frac{\sin^2\alpha}{2\cos\alpha}$ = critical collision frequency.

We will consider a plane wave propagating in the plane of the magnetic meridian in a horizontally stratified ionosphere.

The angle of incidence (θ_o), the angle of the wave normal in the layer of index of refraction n (θ), and the angle of the magnetic field (α) in the selected co-ordinate system are shown in Fig. 1.

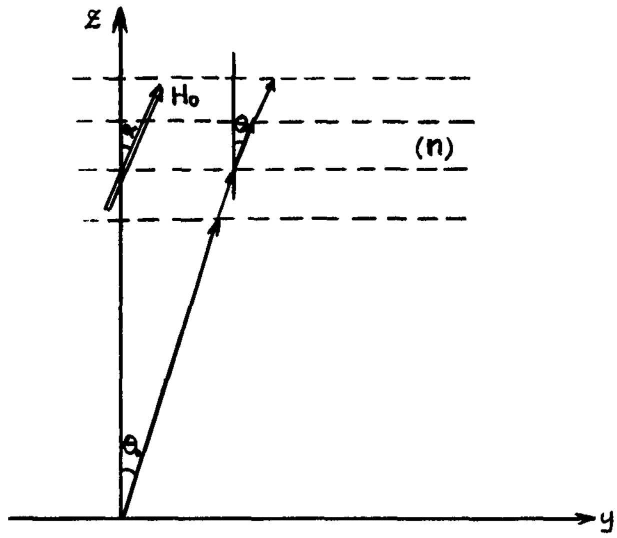

Figure 1

The following compact symbols will also be used.

$$X = \frac{f_N^2}{f^2} = \frac{Ne^2}{\pi mf^2}$$

$$Y = \frac{f_N}{f} = \frac{eH_o}{2\pi mcf}, \quad Y_L = Y\cos\alpha \ , \quad Y_T = Y\sin\alpha$$

$$U = 1-iZ = 1-i\,\frac{\nu}{2\pi f} \quad (\text{i.e., } Z = \frac{\nu}{2\pi f})$$

$$\cos\theta_o = C, \quad \sin\theta_o = S$$

<u>Development of the basic equations</u>

The equation of motion of a plane wave in a magnetoionic medium is:

$$m\ddot{\vec{r}} + m\nu\dot{\vec{r}} = e\vec{E} + \frac{e}{c}\,\dot{\vec{r}}\wedge\vec{H}_o. \qquad [1]$$

Introducing the volume polarization $\vec{P} = \vec{P}_o e^{i\omega t} = Ne\vec{r}$ we obtain

$$\frac{-m\omega^2}{Ne}\vec{P} + \frac{im\nu\omega}{Ne}\vec{P} = e\vec{E} + \frac{e}{c}\frac{i\omega}{Ne}\vec{P}_\wedge\vec{H}_o$$

and, therefore

$$\frac{\vec{E}}{4\pi} = \frac{-m\omega^2}{4\pi Ne^2}(1-i\frac{\nu}{\omega})\vec{P} + \frac{im\omega^2}{4\pi Ne^2}\frac{1}{\omega}\frac{e\vec{H}_o}{mc}\wedge\vec{P}$$

or

$$\frac{-X}{4\pi}\vec{E} = U\vec{P} - i\vec{Y}_\wedge\vec{P} \qquad [2]$$

writing eqn. [2] in matrix form we have

$$-\frac{X}{4\pi}\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \begin{bmatrix} U & iY_L & -iY_T \\ -iY_L & U & 0 \\ iY_T & 0 & U \end{bmatrix}\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$

and inverting the matrix equation, provided $U^2 \neq Y^2$, we obtain:

$$\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \frac{-X}{4\pi U(U^2-Y^2)}\begin{bmatrix} U^2 & -iY_LU & iY_TU \\ iY_LU & U^2-Y_T^2 & -Y_LY_T \\ -iY_TU & -Y_LY_T & U^2-Y_L^2 \end{bmatrix}\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \qquad [3]$$

The matrix eqn. [3] can be written as:

$$P = \alpha E$$

where matrix α is called the susceptibility matrix. We also have:

$D = \epsilon E = E + 4\pi P = E + 4\pi\alpha E = (1 + 4\pi\alpha)\, E$ and thus:

$$\epsilon = 1 + 4\pi\alpha \qquad [4]$$

where ϵ is now the complex dielectric tensor.

From eqns. [3] and [4] we obtain the expression for ϵ :

$$\begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} = \begin{bmatrix} 1 - \frac{XU^2}{U(U^2-Y^2)} & \frac{iXY_LU}{U(U^2-Y^2)} & -i\frac{XY_TU}{U(U^2-Y^2)} \\ -i\frac{XY_LU}{U(U^2-Y^2)} & 1 - \frac{X(U^2-Y_T^2)}{U(U^2-Y^2)} & \frac{XY_TY_L}{U(U^2-Y^2)} \\ \frac{iXY_TU}{U(U^2-Y^2)} & \frac{XY_TY_L}{U(U^2-Y^2)} & 1 - \frac{X(U^2-Y_L^2)}{U(U^2-Y^2)} \end{bmatrix} \qquad [5]$$

The dependence of the wave functions for propagation in the y-z plane through a horizontally stratified ionosphere (along the z axis) is:

$$F = F_o\, e^{-ikSy + i\omega t},$$

where

$$k = \frac{2\pi}{\lambda} = \frac{2\pi f}{c} = \frac{\omega}{c}$$

so that we have:

$$\frac{d}{dt} = i\omega, \quad \frac{d}{dx} = 0, \quad \frac{d}{dy} = -ikS.$$

Let us also introduce the operator $-\frac{1}{ik}\frac{d}{dz} \equiv q$.

Maxwell's equations in the absence of source currents and changes become:

$$\nabla_\wedge E = -\frac{1}{c}\dot{H} = -\frac{i\omega}{c} H \qquad [6]$$

$$\nabla_\wedge H = \frac{1}{c}\dot{D} = \frac{i\omega}{c} D = \frac{i\omega}{c}\epsilon E \qquad [7]$$

Using the operators defined above, eqns. [6] and [7] take the form of eqns. [8] and [9] respectively.

$$\begin{aligned} -q\,E_y + SE_z &= H_x \\ -q\,E_x &= -H_y \\ SE_x &= -H_z \end{aligned} \qquad [8]$$

$$\begin{aligned} -q\,H_y + SH_z &= -\epsilon_{xx}E_x - \epsilon_{xy}E_y - \epsilon_{xz}E_z \\ -q\,H_x &= \epsilon_{yx}E_x + \epsilon_{yy}E_y + \epsilon_{yz}E_z \\ SH_x &= \epsilon_{zx}E_x + \epsilon_{zy}E_y + \epsilon_{zz}E_z \end{aligned} \qquad [9]$$

Eliminating E_z and H_z from eqns. [8] and [9] we obtain a system of four equations:

$$-q\,E_x = -\,H_y$$

$$-q\,E_y = S\,\frac{\epsilon_{zx}}{\epsilon_{zz}}\,E_x + S\,\frac{\epsilon_{zy}}{\epsilon_{zz}}\,E_y + \left(1-\frac{S^2}{\epsilon_{zz}}\right) H_x$$

$$-q\,H_x = \left(\epsilon_{yx} - \frac{\epsilon_{yz}\,\epsilon_{zx}}{\epsilon_{zz}}\right) E_x + \left(\epsilon_{yy} - \frac{\epsilon_{yz}\,\epsilon_{zy}}{\epsilon_{zz}}\right) E_y + S\,\frac{\epsilon_{yz}}{zz}\,H_x \qquad [10]$$

$$-q\,H_y = \left(-\epsilon_{xx} + \frac{\epsilon_{xz}\,\epsilon_{zx}}{\epsilon_{zz}} + S^2\right) E_x + \left(-\epsilon_{xy} + \frac{\epsilon_{xz}\,\epsilon_{zy}}{\epsilon_{zz}}\right) E_y - S\,\frac{\epsilon_{xz}}{\epsilon_{zz}}\,H_x$$

Substituting from eqn. [5] the ϵ tensor elements in eqn. [10] and writing it also in a matrix form we have

$$\begin{bmatrix} q & 0 & 0 & 1 \\ -i\,\dfrac{XY_TUS}{U(U^2-Y^2)-X(U^2-Y_L^2)} & \dfrac{XY_LY_TS}{U(U^2-Y^2)-X(U^2-Y_L^2)}+q & \dfrac{C^2U(U^2-Y^2)-X(U^2-Y_L^2)}{U(U^2-Y^2)-X(U^2-Y_L^2)} & 0 \\ \dfrac{iXY_L(U-X)}{U(U^2-Y^2)-X(U^2-Y_L^2)} & \dfrac{U(U-X)^2-Y^2(U-X)}{U(U^2-Y^2)-X(U^2-Y_L^2)} & \dfrac{XY_LY_TS}{U(U^2-Y^2)-X(U^2-Y_L^2)}+q & 0 \\ C^2-\dfrac{XU(U-X)}{U(U^2-Y^2)-X(U^2-Y_L^2)} & \dfrac{-iXY_L(U-X)}{U(U^2-Y^2)-X(U^2-Y_L^2)} & i\,\dfrac{XY_TUS}{U(U^2-Y^2)-X(U^2-Y_L^2)} & q \end{bmatrix} \begin{bmatrix} -E_x \\ E_y \\ H_x \\ H_y \end{bmatrix} = 0 \qquad [11]$$

For the case of vertical incidence i.e., $S = 0$, $C = 1$, eqn. [11] takes the form:

$$\begin{bmatrix} q & 0 & 0 & 1 \\ 0 & q & 1 & 0 \\ \dfrac{iXY_L(U-X)}{U(U^2-Y^2)-X(U^2-Y_L^2)} & \dfrac{U(U-X)^2-Y^2(U-X)}{U(U^2-Y^2)-X(U^2-Y_L^2)} & q & 0 \\ \dfrac{U(U-X)^2-UY^2+XY_L^2}{U(U^2-Y^2)-X(U^2-Y_L^2)} & \dfrac{-iXY_L(U-X)}{U(U^2-Y^2)-X(U^2-Y_L^2)} & 0 & q \end{bmatrix} \begin{bmatrix} -E_x \\ E_y \\ H_x \\ H_y \end{bmatrix} = 0 \qquad [12]$$

Finally we can eliminate H_x and H_y from eqn. [12] and we have:

$$\begin{bmatrix} q^2 - \dfrac{U(U-X)^2-UY^2+XY_L^2}{U(U^2-Y^2)-X(U^2-Y_L^2)} & -i\dfrac{XY_L(U-X)}{U(U^2-Y^2)-X(U^2-Y_L^2)} \\ i\dfrac{XY_L(U-X)}{U(U^2-Y^2)-X(U^2-Y_L^2)} & q^2 - \dfrac{U(U-X)^2-Y^2(U-X)}{U(U^2-Y^2)-X(U^2-Y_L^2)} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = 0 \qquad [13]$$

Eqns. [11] and [13] are respectively the basic equations governing the propagation of electromagnetic waves in magnetoionic media for oblique and vertical incidence.

The Booker's quartic and its solutions

For a homogeneous region where X, Y, U are constants, the z dependence is only in the exponent of the wave function, i.e.,

$$qF = \frac{-i}{ik}\frac{d}{dz}F_o e^{-ikn\cos\theta z} = \frac{-ikn\cos\theta}{-ik}F_o e^{-ikn\cos\theta z} = n\cos\theta_o\, F_o e^{-ikn\cos\theta} = n\cos\theta F$$

and therefore

$$q = n\cos\theta.$$

Requiring the determinant of eqn. [11] to be zero we obtain a quartic equation in q called the Booker's quartic.

$$F(q) = \alpha q^4 + \beta q^3 + \gamma q^2 + \delta q + \epsilon = 0 \qquad [14]$$

where

$$\alpha = U(U^2-Y^2)-X(U^2-Y_L^2) = U^2(U-X)-UY^2 + XY_L^2$$

$$\beta = 2SXY_LY_T$$

$$\gamma = -2U[(C^2U-X)\ (U-X)-C^2Y^2]-X[Y^2-S^2Y_T^2 + C^2Y_L^2]$$

$$\delta = -2SC^2XY_LY_T = -C^2\beta$$

$$\epsilon = (C^2U-X)\ [(C^2U-X)(U-X)-C^2Y^2]-S^2C^2Y_T^2X.$$

For vertical incidence ($S = 0$ and $C = 1$), β and δ become zero and thus we have:

$$F(q) = \alpha q^4 + \gamma q^2 + \epsilon = 0 \qquad [15]$$

where

$$\alpha = U^2(U-X)-UY^2 + XY_L^2$$

$$\gamma = -2U[(U-X)^2-Y^2]-X(Y^2 + Y_L^2)$$

$$\epsilon = (U-X)[(U-X)^2-Y^2]$$

Eqn. [15] is also obtained by equating to zero the determinant of eqn. [13]. Thus, for vertical incidence, Booker's quartic becomes a biquadratic $\alpha q^4 + \gamma q^2 + \epsilon = 0$ which is easily solved to give the Appleton-Hartree equation:

$$q^2 = 1 - \frac{X}{U - \frac{Y_T^2}{2(U-X)} \pm \sqrt{\frac{Y_T^4}{4(U-X)^2} + Y_L^2}} \qquad [16]$$

where now, since $\cos\theta = 1$, $q^2 = n^2$ = the complex index of refraction.

When collisions are neglected ($\nu \ll \omega$), so that $U = 1$, the solutions of the Appleton-Hartree equation for $Y < 1$, are:

$n = 0$ at $X = 1$ (ordinary)

$n = 0$ at $X = 1 - Y$ (extraordinary)

we also have an infinity, $n = \infty$, at $X = \frac{1 - Y^2}{1 - Y_L^2}$, for the extraordinary. These solutions hold for all values of the angle α, with one exception, the case when $\alpha = 0$, i.e., $Y_T = 0$ $Y_L = Y$. Then

$$n^2 = 1 - \frac{X}{1 \pm Y}$$

and the roots are:

$n = 0$ at $X = 1 + Y$ (ordinary)

$n = 0$ at $X = 1 - Y$ (extraordinary)

and infinity ($n = \infty$) at $X = 1$ for the extraordinary.

We see, therefore, that at vertical incidence, the extraordinary wave will always be reflected (for $Y < 1$) at $X = 1 - Y$. An exponentially decaying part of it (evanescent wave) can advance farther, but it will be absorbed at $X = \frac{1 - Y^2}{1 - Y_L^2}$ where $n \rightarrow \infty$.

The ordinary wave in the case of vertical incidence will always be reflected at $X = 1$ except when the magnetic field is also vertical, in which case it will be reflected at $X = 1 + Y$. For the more general case of oblique incidence in the plane of the magnetic meridian the four roots of Booker's quartic correspond to the four waves:

1) Ascending ordinary
2) Descending ordinary
3) Ascending extraordinary
4) Descending extraordinary

When two of the roots are equal, we have either reflection, e.g., roots of ascending and descending ordinary waves equal, or coupling e.g., roots of ascending ordinary and extraordinary equal.

To have two roots of eqn. [14] equal we must also have

$$\frac{dF(q)}{dq} = 4\alpha q^3 + 3\beta q^2 + 2\gamma q + \delta = 0. \qquad [17]$$

In order to satisfy eqns. [14] and [17] simultaneously we must have

$$\begin{bmatrix} 8\alpha\gamma - 3\beta^2 & 6\alpha\delta - \beta\gamma & 16\alpha\epsilon - \beta\delta \\ 6\alpha\delta - \beta\gamma & 4\alpha\epsilon - \gamma^2 + 2\beta\delta & 6\epsilon\beta - \gamma\delta \\ 16\alpha\epsilon - \beta\delta & 6\beta\epsilon - \gamma\delta & 8\epsilon\gamma - 3\delta^2 \end{bmatrix} = 0$$

The solutions of this determinant have been investigated by M. L. V. Pitteway (1959). In the simpler case of vertical incidence we simply have $\beta \equiv \delta \equiv 0$, and the above determinant reduces to:

$$\alpha\epsilon(\gamma^2 - 4\alpha\epsilon)^2 = 0,$$

which is satisfied when:

$\underline{\alpha = 0}$: $(1-X)-Y^2 + XY_L^2 = 0$ i.e., $X = \dfrac{1 - Y^2}{1 - Y_L^2}$ (Infinity)

$\underline{\epsilon = 0}$: $(1-X)[(1-X)^2 - Y^2] = 0$ i.e., $X = 1$, $X = 1 - Y$, $X = 1 + Y$ (Reflections)

$\underline{\gamma^2 - 4\alpha\epsilon = 0}$: $X(U-X)\sqrt{\dfrac{Y_T^4}{4(U-X)^2} + Y_L^2} = 0$ which at $X = 1$ yields:

$$Z = \frac{Y_T^2}{2Y_L} \quad \text{i.e.,} \quad \nu_c = 2\pi f_H \frac{\sin^2\alpha}{2\cos\alpha} \quad \text{(Coupling)}$$

Diagrammatic representation

We will now investigate the variation of q with X in eqns. [14] and [15]

$$\frac{dq}{dX} = - \frac{\frac{\partial\alpha}{dX}q^4 + \frac{\partial\beta}{dX}q^3 + \frac{\partial\gamma}{dX}q^2 + \frac{\partial\delta}{dX}q + \frac{\partial\epsilon}{dX}}{\frac{dF}{dq}} \qquad [18]$$

$\frac{dF}{dq} \equiv 0$ at the point where F(q) has a double root. At that point, $\frac{dq}{dX}$ becomes infinite

and the q-X curves have a vertical tangent. If, however, the numerator of eqn. [18] is zero also at the same point, $\frac{dq}{dX}$ is not infinite and the tangent is not vertical F(q), neglecting collisions, can be factored at X = 1 to give:

$$F(q) = (q^2-C^2)(q-S\frac{Y_L}{Y_T})^2 = 0.$$

Coupling will take place at the double root, i.e., at $q = S\frac{Y_L}{Y_T}$. Inserting in the numerator of eqn. [18] the above value of q and the derivatives:

$$\left[\frac{d\alpha}{dX}\right]_{X=1} = Y_L^2-1$$

$$\left[\frac{d\beta}{dX}\right]_{X=1} = 2SY_LY_T$$

$$\left[\frac{d\gamma}{dX}\right]_{X=1} = -2S^2-Y^2 + S^2Y_T^2-C^2Y_L^2$$

$$\left[\frac{d\delta}{dX}\right]_{X=1} = -2SC^2Y_LY_T$$

$$\left[\frac{d\epsilon}{dX}\right]_{X=1} = -S^4 + C^2Y^2 - S^2C^2Y_T^2.$$

We obtain:

$$\frac{Y^2(S^2-\sin^2\alpha)^2-S^4}{\sin^4\alpha},$$

which becomes equal to zero when:

$$S = \sin\theta_o = \pm\sqrt{\frac{Y}{Y\pm 1}}\sin\alpha. \qquad [19]$$

Thus, at X = 1, for $Y < 1$ and $\sin\theta_o = \pm\sqrt{\frac{Y}{Y+1}}\sin\alpha$, we have complete coupling instead of reflection. No collisions are necessary for this coupling. For the

case of vertical incidence ($\theta_0 = 0$) complete coupling without collisions occurs only when $\alpha = 0$. When $\alpha \neq 0$, coupling occurs at X = 1 only in the presence of the required collision frequency

$$\nu = \nu_c = 2\pi f_H \frac{\sin^2\alpha}{2\cos\alpha}.$$

No such simple formula exists for the case of oblique incidence when $\sin\theta_0 \neq \pm\sqrt{\frac{Y}{Y+1}}$ si

In Fig. 2 we will plot the curves of q vs. X for vertical and oblique incidence, illustrating schematically the regions of reflection and coupling. The curves are derived from Booker's quartic, neglecting the collision frequency, i.e., U = 1.

For the case of vertical incidence ($\theta_0 = 0$) in Fig. 2, we show the following situations:

(a) The magnetic field makes a rather large angle (α) with the vertical. Reflections: Ordinary at X = 1, Extraordinary at X = 1 - Y.

(b) Angle $\alpha \simeq 0$. Here we have coupling between the ordinary and extraordinary waves, producing the Z-mode which is reflected at X = 1 + Y. The shaded areas show the regions of coupling.

(c) Angle $\alpha = 0$. We no longer have the ordinary wave reflected at X = 1.

Note that the vertical incidence curves are symmetric as the ascending and descending waves travel the same path. For the case of oblique incidence we have correspondingly:

(d) The dashed lines represent the case where θ_0 is much larger than α. The reflections are no longer at X = 1 - Y, 1, and 1 + Y but at

VERTICAL INCIDENCE　　　　OBLIQUE INCIDENCE

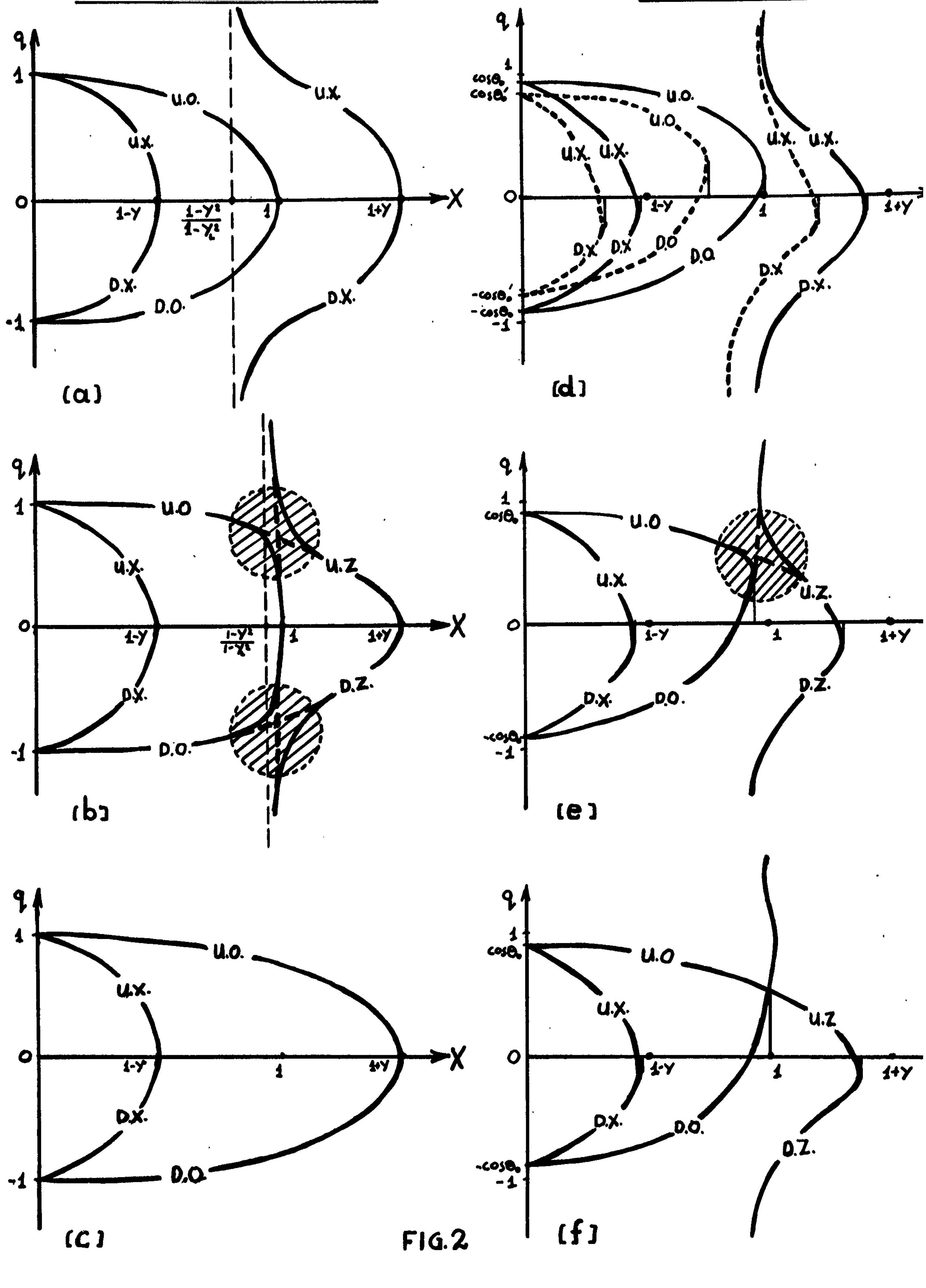

FIG.2

lower values of X, given respectively to a first approximation by $X = (1 - Y)\cos\theta_0$, $\cos\theta_0$, and $(1 + Y)\cos\theta_0$. The solid lines represent the case where $\sin\theta_0 < \sqrt{\frac{Y}{Y+1}}\sin\alpha$. Here θ_0 is quite small and the curves approach those of vertical incidence. For all the values of $0 < \theta < \sin^{-1}\left\{\sqrt{\frac{Y}{Y+1}}\sin\alpha\right\}$, the ordinary is reflected at $X = 1$.

(e) Here $\sin\theta_0 \simeq \sqrt{\frac{Y}{Y+1}}\sin\alpha$ and we have coupling, i.e., part of the ordinary wave is reflected near $X = 1$ and part of the wave couples to the extraordinary which propagates farther and is finally reflected near $X = (1 + Y)\cos\theta_0$.

(f) This is the case where $\sin\theta_0 = \sqrt{\frac{Y}{Y+1}}\sin\alpha$. At $X = 1$ we now have complete coupling instead of reflection. We notice that the oblique incidence curves are not symmetric because the descending path is different from the ascending one.

When a moderate number of collisions are present, e.g., $\frac{\nu}{\omega} \simeq 0.01$, the curves are very similar to those shown with the exception of the coupling regions where a full wave treatment is necessary.

The coupling parameters have been determined for the case of vertical incidence through the full wave treatment of the problem, but no similar analysis has yet been accomplished for the case of oblique incidence where the physical mechanism is analogous but the mathematics are very much more complex.

Coupling

We will now examine the coupoing process for the case of vertical incidence. An ordinary wave incident normally on an interface separating two regions of different indices of refraction ($n_o = 1 - \frac{X}{U-iY_L\rho_o}$) and accordingly different polarizations (ρ_o), will have to produce an extraordinary wave in order to satisfy the boundary conditions.

The extraordinary wave produced will increase as the change of ordinary polarization ($\Delta\rho_o$) increases, or as the difference between the ordinary (ρ_o) and extraordinary (ρ_x) polarizations decreases.

A continuously varying medium can be thought of as a series of thin slabs of different n_o. Extraordinary waves are produced at each interface and, in the limit, the coupling region may be thought of as a distributed source of upgoing extraordinary waves. When ρ_o-ρ_x is very small, the extraordinary waves will all have nearly the same phase and will therefore add constructively.

A coupling parameter, ψ , is defined proportional to $\frac{d\rho_o}{dz}$ and $\frac{1}{\rho_o-\rho_x}$ in accordance with the above-mentioned dependencies. Using the fact that $\rho_o\rho_x = 1$ we have:

$$\psi = \frac{1}{\rho_o^2 - 1}\frac{d\rho_o}{dz} = \frac{1}{2}\frac{d}{dz}\ln\frac{\rho_o - 1}{\rho_o + 1} . \qquad [20]$$

From the Appleton-Hartree equation, the polarization ρ_o is given as:

$$\rho_o = \frac{-i}{Y_L}\left\{\frac{Y_T^2}{2(U-X)} - \sqrt{\frac{Y_T^4}{U(U-X)} + Y_L^2}\right\} . \qquad [21]$$

Substituting eqn. [21] into eqn. [20] we obtain:

$$\psi = \frac{-iY_T^2\, Y_L}{4(U-X)^2\, Y_L^2 + Y_T^4}\, \frac{dX}{dz}. \qquad [22]$$

The greater the coupling parameter ψ , the larger is the extraordinary wave generated in the coupling region.

At X = 1, the denominator of eqn. [22] becomes zero, i.e., $\psi = \infty$ (complete coupling) at

$$-4\, \frac{\nu_c^2}{\omega^2}\, Y_L^2 + Y_T^4 = 0;$$

from which the critical collision frequency ν_c is derived

$$\nu_c = \frac{\sin^2\alpha}{2\cos\alpha}\, 2\pi\, f_H. \qquad [23]$$

Using a dipole approximation for the earth's magnetic field, we have:

$$H_o = \frac{M}{r^3}\,(1 + 3\sin^2\phi)^{1/2}$$

$$f_H = 2.8\; H_o;$$

$$\tan\alpha = 1/2\ \cot\phi.$$

Using the above relations, in eqn. [23] we have:

$$\nu_c \simeq 1.4\, \frac{\cos^2\phi}{\sin\phi}$$

which is used in Fig. 3 to plot ν_c vs. the geomagnetic latitude ϕ.

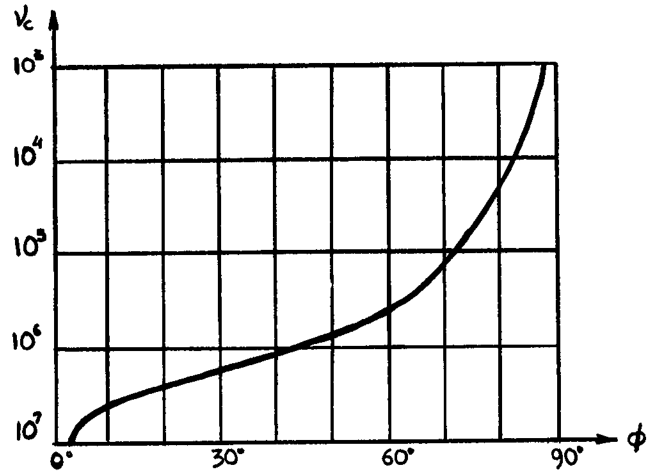

Figure 3

Transmission coefficient (T_{oz})

Let us define ϵ such that

$$X = \frac{e^2 N}{\pi m f^2} = 1 + \epsilon z.$$

Close to the ordinary reflection level, $X \simeq 1$ (i.e., $\epsilon z \ll 1$) and we can utilize the expansion:

$$X = X_o + z\left(\frac{dX}{dz}\right)_o = 1 + z\,\frac{e^2}{\pi m f^2}\left(\frac{dN}{dz}\right)_o = 1 + z\,\frac{1}{N_o}\left(\frac{dN}{dz}\right)_o = 1 + \left(\frac{1}{N}\frac{dN}{dz}\right)_o z$$

from which it follows that:

$$\left(\frac{dX}{dz}\right)_o = \epsilon = \left(\frac{1}{N}\frac{dN}{dz}\right)_o.$$

The transmission coefficient for vertical incidence was first calculated by Rydbeck (1950). V. L. Ginzburg (1961) in his book, pp. 515-559, derives it in three

different ways. We will avoid repeating the derivations, giving only the final results.

$$|T_{oz}| = e^{-\delta} \quad \text{where} \quad \delta = \frac{\nu_c \beta_o}{c} \left(1 - \frac{\nu}{\nu_c}\right)^{3/2} \qquad [24]$$

and

$$\nu_c = 2\pi f_H \frac{\sin^2\alpha}{2\cos\alpha}, \qquad \beta_o = \frac{\pi}{2} \frac{\cos\alpha \sqrt{Y(Y+1)}}{(1 + Y\cos\alpha)^2}, \qquad \epsilon = \left(\frac{1}{N}\frac{dN}{dz}\right)_o ,$$

$$c = 3 \times 10^{10} \text{cm/sec.}$$

For a parabolic layer, where $N = N_{max}\left[1 - \left(\frac{z_o - z}{\Delta h}\right)^2\right]$, we have

$$f_N^2 = \frac{Ne^2}{\pi m} = \frac{N_{max}\, e^2}{\pi m}\left[1 - \frac{z_o - z}{h}^2\right] = f_{Nmax}^2\left[1 - \frac{z_o - z}{h}^2\right].$$

An ordinary wave of frequency f will be reflected at $X = 1$, i.e., $f_N = f$, which as seen from the above equation, will correspond to a height

$$\left(\frac{z_o - z}{\Delta h}\right)_o = \sqrt{1 - \frac{f^2}{f_{Nmax}^2}}.$$

For a parabolic layer therefore:

$$\epsilon = \frac{1}{N_o}\left(\frac{dN}{dz}\right)_o = \frac{e^2}{\pi m f^2} N_{max}\, 2\left(\frac{z_o - z}{\Delta h}\right)_o \frac{1}{\Delta h} = \frac{f_{Nmax}^2}{f^2}\, 2\sqrt{1 - \frac{f^2}{f_{Nmax}^2}}\, \frac{1}{\Delta h}$$

or

$$\epsilon = \frac{2}{\Delta h}\;\frac{f^2_{Nmax}}{f^2}\;\sqrt{1 - \frac{f^2}{f^2_{Nmax}}}\;.$$

Thus, for $\nu < \nu_c$ the complete expression for the Z-mode transmission coefficient assuming a parabolic layer becomes:

$$\left|T_{oz}\right| = \exp\left\{ -\frac{\pi\Delta h}{4C}\;\frac{\cos\alpha\sqrt{Y(Y+1)}}{(1 + Y\cos\alpha)^2}\;\frac{f^2/f^2_{Nmax}}{\sqrt{1 - f^2/f^2_{Nmax}}}\;\nu_c\;(1 - \nu/\nu_c)^{3/2}\right\}. \qquad [25]$$

When $\nu \gg \nu_c$, T_{oz} becomes unity.

Fig. 4 shows $\left|T_{oz}\right|^2$ vs. ν/ν_c for the conditions Δh = 30 km, $\alpha = 13^\circ$ (~geom. lat. 65°), f_H = 1.44 Mc, f = 3.47 Mc, f_{Nmax} = 4 Mc, and ν_c = 0.23 Mc

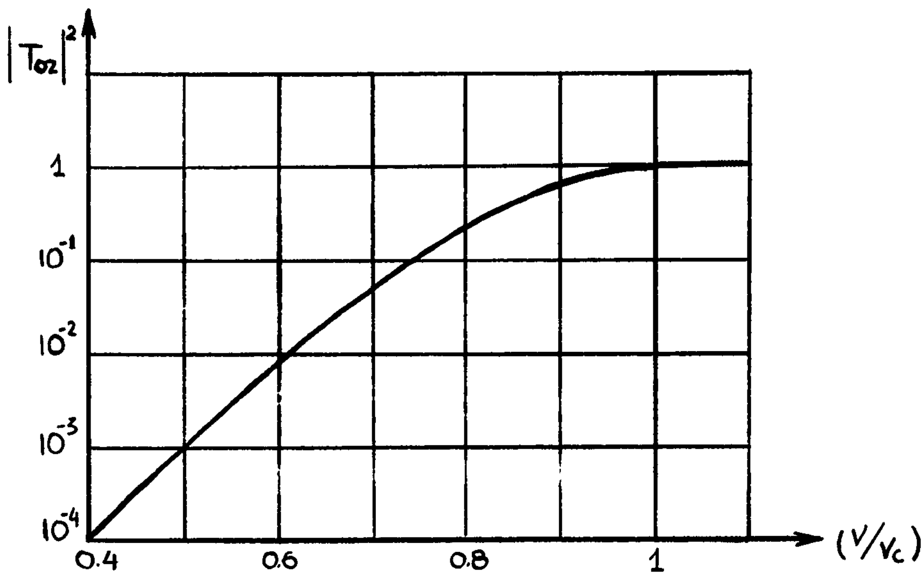

Figure 4

From Figs. 3 and 4 we see that only for high latitudes (preferably above 70°N or below 70°S) can we have E-layer coupling. For F-layer coupling where $(\nu/\nu_c) \rightarrow 0$, even higher latitudes are required.

For example, for $f_{Nmax} = 2f = 4f_H$, $\Delta h \simeq 40$ km and $(\nu/\nu_c) \rightarrow 0$, we have $|T_{oz}|^2 = \exp(-200 \sin^2\alpha)$ which means that at $\alpha \simeq 9^{\circ}$, i.e., geomagnetic latitude $\phi \simeq 73^{\circ}$, $|T_{oz}|^2 \simeq 10^{-2}$. At Boston where $\phi \simeq 53^{\circ}$ and $\alpha \simeq 21^{\circ}$ $|T_{oz}|^2 \simeq 10^{-11}$. So we see that we can have vertical coupling only at high geomagnetic latitudes in the E-region and only at very high geomagnetic latitudes in the F-region.

Oblique incidence

For oblique incidence we have complete transmission when $\sin\theta_o = \sqrt{\frac{Y}{Y+1}} \sin\alpha$ (as we have already seen from eqn. [19]), with a collision frequency of zero as for vertical incidence, when $\alpha = 0$. If collisions are present they will broaden, through coupling, the tolerance in the angle θ_o allowing a cone rather than a single direction of propagation in the Z-mode. It is not proper to calculate this cone, as some authors have done, by utilizing the critical collision frequency for vertical incidence $\nu_c = \dfrac{2\pi f_H \sin^2\alpha}{2\cos\alpha}$, as the formula is not valid for oblique incidence.

The problem when considering oblique incidence is not the penetration of the layer by the Z-mode, but the return path. One has to assume that the reflecting layer is rough enough to produce back scattering, so that some extraordinary wave can return along the incident path (parallel to the magnetic field).

The observations of Ellis support this theory which can explain the existence of the Z-mode at moderate latitudes, where, as it is also pointed out by Budden (1961),

the explanation through vertical coupling would be very difficult. Some doubts, however, still exist, Ratcliffe (1959), about the ability of the reflection layer to produce sufficient back scattering.

Fig. 5 demonstrates the generation of the Z-mode at oblique incidence through back scattering.

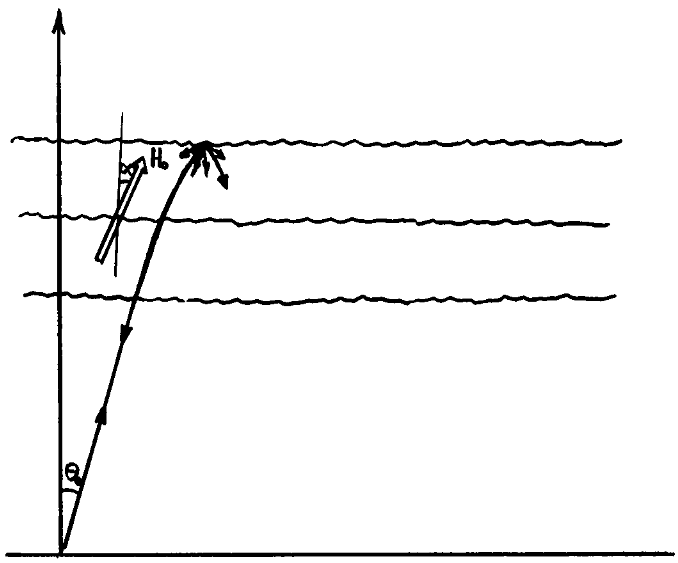

Figure 5

It is quite clear that the larger the angle α, i.e., the lower the geomagnetic latitude, the more difficult it is to have sufficient back scattering in the direction of the magnetic field. In addition, the ionosonde transmitters used were primarily for vertical incidence, so that the intensity of the obliquely incident wave was much weaker. These reasons explain why the Z-mode has not been observed at low latitudes where θ_0 and α must be quite large.

Previous work

It is of interest to give an account of the work done to date on the Z-mode, discussing at the same time some of the ideas put forward in the light of today's accepted theories.

T. L. Eckersley (1933), in a discussion of the ionosphere, said that on his wholly visual observation, of the ionospheric reflections on a cathode ray oscillograph, he noticed that "On rare occasions there are very definite triplets, with one right hand and two left hand following, all sufficiently close to ensure that they belong to the same system. Triplets have been observed on five occasions during the last five months on wavelength between 55 and 60 meters."

G. R. Toshniwal (1935) reported that in India, working on a single frequency, they observ ed mostly after sunset triplets lasting for not more than a minute.

Toshniwal still uses the Lorentz term in the equations, and thinks that an evanescent extraordinary wave partially penetrates and reaches the other reflection point, which Mary Taylor (1933) had suggested as a possibility if the electron density varies quite rapidly with height.

Leiv Harang (1936) gave the first records of ionograms showing a Z-mode. This is also called triple splitting because the ionograms have the form shown in Fig. 6.

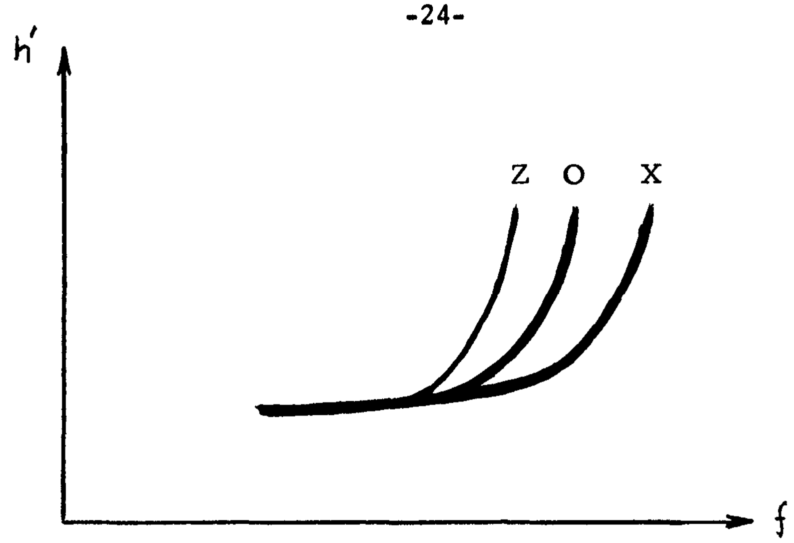

Figure 6

He observed the Z-mode during winter at Tromsö, Norway, only during the day and usually around noon when f_oF_2 was high. He also believes he is observing two extraordinary and one ordinary as Mary Taylor had suggested.

Since that time many other people have observed the Z-mode and have published records of triple splitting from different parts of the world.

D. H. Meek (1948) reported that the Z-mode had been observed in Canada since 1943. He reports that it was often seen extending from the E-region through the F_1 region to the F_2 region.

From a sample record accompanying his report he reads:

$$f_zF_2 = 5.45 \text{ Mc}, \quad f_oF_2 = 6.10 \text{ Mc}, \quad f_xF_2 = 6.95 \text{ Mc}.$$

From the basic equations we have X = 1 - Y, 1,1+Y, i.e.,

$$f_N^2 = f_o^2 = f_x^2 - f_xf_H = f_z^2 + f_zf_H,$$

solving the above relations for f_H we obtain

$$f_H = (f_x-f_z) = 2(f_x-f_o) - \frac{(f_x-f_o)^2}{f_x} = 2(f_o-f_z) + \frac{(f_o-f_z)^2}{f_z}$$

which yields respectively

$$f_H = 1.50 \text{ Mc}, \quad 1.60 \text{ Mc}, \quad 1.38 \text{ Mc}.$$

He has not given an explanation for those differences. On a qualitative basis, though, they can be explained as follows. The ordinary and extraordinary waves are deflected, the first toward the North Pole where the critical frequency is lower, and the second toward the equator where the critical frequency is higher. Thus (f_x-f_o) is larger than it would have been if both were reflected at the same point, and thus we get a larger value of f_H.

The critical frequency of the Z-mode, however, is slightly higher because of the oblique incidence, i.e., $(f_z)_{obl.} \simeq \frac{(f_z)_{norm}}{\cos\theta_o}$. Thus (f_o-f_z) is smaller and gives a smaller value for f_H. Finally, in the case (f_x-f_z), both are increased and thus f_H has an intermediate value probably closer to its actual one.

Triple splitting was observed also by Gordon Newshead (1948) at Hobart, Tasmania, during 1947. Only 39 clear triple splitting records out of 5000 were identified, occurring mostly between the hours of 1700 and 2000 local time.

In a record presented he gives:

$$f_x = 8.5 \text{ Mc}, \quad f_o = 7.6 \text{ Mc}, \quad f_z = 7 \text{ Mc}.$$

He states that f_z is higher than anticipated and claims that this probably can be

accounted for, if collisions had been included in the calculations. Though collisions might play some role, the explanation seems basically to be the oblique incidence of the Z-mode. He still speculates that the Z-mode might be the effect that Mary Taylor suggested in 1933.

T. L. Eckersley (1948), commenting on Newshead's observations, refers to his original observations (1933) where two of the three received waves had ordinary and only one had extraordinary polarization. If Mary Taylor's scheme was the explanation, we would have two waves with extraordinary and only one with ordinary polarization. He concludes that the explanation of the Z-mode is the vertical coupling of the ordinary and the extraordinary waves and not the partial penetration of an evanescent extraordinary wave as Mary Taylor had suggested.

The Z-trace polarization was accurately measured by J. E. Hogarth (1951) at Fort Chimo in Northern Canada. It was found to be ordinary, i.e., in the northern hemisphere, anticlockwise as viewed in the direction of propagation. B. Landmark (1952), working under L. Harang in Tromsö, Norway, repeated the measurements and his results agreed completely with those of Hogarth. Thus it was clearly established that a coupling process was responsible for the Z-mode.

O. E. H. Rydbeck (1950) derived his expression for the transmission coefficient due to coupling at vertical incidence. This theory can explain coupling only in the E-region and only for high latitudes as we have already seen.

Eckersley (1950) also treated the coupling phenomenon, but from a more general point of view, using Rieman surfaces.

Meanwhile, James C. W. Scott (1950) advocated the new idea that the Z-mode was produced by propagation at oblique incidence along the magnetic field and back scattering from a rough reflecting layer. He implied though that the angle of incidence should be that of the magnetic field and that, if $\nu \geq \nu_c = \frac{2\pi f_H \sin^2\alpha}{2\cos\alpha}$, we have longitudinal propagation which holds for vertical incidence but not necessarily for oblique incidence, as is the case here.

G. R. Ellis (1953) developed more carefully the theory of the Z-mode suggested by Scott, and derived the correct expression for the angle of incidence, θ_0, for Z-mode propagation; i.e.,

$$\sin\theta_0 = \sqrt{\frac{Y}{Y+1}}\,\sin\ .$$

He made the first measurements of θ_0 in Hobart, Tasmania ($\alpha \sim 18^{\circ}$) using two spaced (120 ft.) loop antennas and found that the propagation was in the plane of the magnetic meridian. His results agreed very well with those predicted from the above formula.

He is still using the vertical incidence equations, however, and therefore his expressions for the cone angle in terms of the critical collision frequency are only approximate.

In a later paper, Ellis (1956) did discuss the theory of the Z-mode propagation on the proper basis of oblique incidence and gave measurements of the Z-mode received intensity at different distances from the transmitter. The results indicate an approximate Gaussian distribution.

In the same article he pointed out that the hole existing in the ionosphere along the direction of the magnetic field for $(f_z)_{obl.} < f < f_o$ could be of use in radio astronomy. For a difference in height of δ-km we have:

$$f_H - f'_H = f_H \left(1 - \frac{3\delta}{R}\right)$$

$$\sin\theta_o = \sqrt{\frac{f_H}{f_H + f}} \sin\alpha$$

and

$$\sin\theta'_o = \sqrt{\frac{f'_H}{f'_H + f}} \sin\alpha \simeq \sqrt{\frac{f_H}{f_H + f}} \left(1 - \frac{3}{2R} \frac{f}{f_H + f}\right)$$

therefore

$$\sin\theta_o - \sin\theta'_o = 2\sin\frac{\theta_o - \theta'_o}{2} \cos\frac{\theta_o + \theta'_o}{2} \simeq (\theta_o - \theta'_o)\cos\theta_o = \frac{3\delta}{2R} \frac{f}{f_H + f} \sqrt{\frac{f_H}{f_H + f}} \sin\alpha =$$

$$= \frac{3}{2R} \frac{f}{f_H + f} \sin\theta_o$$

and finally

$$\Delta\theta = \theta_o - \theta'_o = \frac{3\delta}{2R} \frac{f}{f_H + f} \tan\theta_o.$$

For values of f between $2f_H$ and $3f_H$ and small θ_o ($\sim 10^o$, i.e., $\alpha \sim 20^o$) we have $\theta_o \simeq 1/6^o$ per 100 km. If, e.g., $f_{Nmax} = f_o = 4$ Mc, then $f_Z \simeq 3.35$ Mc. Operating at $f = 3.5$ Mc we have

$$\frac{N}{N_{max}} = \left(\frac{f}{f_o}\right)^2 \simeq \frac{3}{4}.$$

For a parabolic layer

$$N = N_{max}\left[1 - \left(\frac{z}{z_{max}}\right)^2\right] \quad \text{and therefore} \quad \frac{3}{4} = \left[1 - \left(\frac{z}{z_{max}}\right)^2\right]$$

or

$$z \simeq \frac{1}{2} z_{max}.$$

That is, the height difference of the two coupling regions, above and below the maximum layer is about half the thickness of the parabolic layer. For the F-region, this would mean more than 100 km which would imply $\theta_o \simeq 0.2^o$. Collisions in this region are very few making uncertain the possibility of attaining this tolerance. Of course one could operate much closer to f_o to reduce θ_o, but attenuation and irregularities then would become much more important.

In a recent article, Ellis (1962) has further developed his previous idea for the application of the Z-mode to radio astronomy in the framework of present day space technology.

He pointed out that a satellite orbiting in a region above the layer of maximum electron density could receive radio waves only through the Z-mode if it operates in a frequency f such that $f_z < f < f_N$ where f_N is the plasma frequency of the environment and $f_z \simeq \frac{1}{\cos\theta_o}\left\{\sqrt{f_N^2 + \left(\frac{f_H}{2}\right)^2} - \frac{f_H}{2}\right\}$. Receiving radio waves only through the Z-mode would render a high resolution to our observations.

This last possibility we will investigate in more detail in the second part of this monograph.

Summary

We have seen the basic equations for propagation of electromagnetic waves in a magnetoionic medium, from which we have deduced that a third wave, called the Z-mode, can propagate under certain conditions beyond the height at which $f^2 = \frac{Ne^2}{\pi m}$. At vertical incidence, the Z-mode will be produced either when the magnetic field is almost vertical (high geomagnetic latitudes) or when the collision frequency ν approaches the critical collision frequency ν_c, i.e., when $\nu \approx \nu_c = 2\pi f_H \frac{\sin^2\alpha}{2\cos\alpha}$. This condition is met only in the lower ionospheric regions, provided we are at high geomagnetic latitudes, so that ν_c is not too large. At mid-latitudes this coupling process is extremely weak and cannot account for the phenomenon of the Z-mode. The oblique incidence approach must therefore be adopted. For this case we have seen that when $\sin\theta_0 = \sqrt{\frac{Y}{Y+1}}\sin\alpha$ the Z-mode penetrates beyond the layer $X = 1$ and is reflected at the layer $X \approx (1 + Y)$. The theory assumes that some back scattering takes place, thus allowing part of the wave to return through the same path. This process does not require any collisions to be present.

No theory for oblique incidence has yet been developed to give the transmission coefficient for angles slightly different from $\theta_0 = \sin^{-1}\left\{\sqrt{\frac{Y}{Y+1}}\sin\alpha\right\}$ or for the dependence of the transmission coefficient on the collision frequency. One can only infer from the existing theory for vertical incidence that the cone angle about θ_0 must be quite small, especially for the very low collision frequencies that prevail

in the upper ionospheric regions. This provides the Z-mode with strong directivity and renders valuable support to Ellis's recent proposal for its use in obtaining high resolution satellite radio astronomical observations at long wavelengths.

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